Duplication of the Sampling Frequency of Periodically Sampled Signals for the Calculation of the Discrete Wigner Distribution*

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A calculation procedure is described which doubles the sampling frequency of periodically sampled signals. Its purpose is to overcome a limitation of the discrete Wigner distribution, which requires a sampling frequency that is twice as high as necessary for ordinary time-signal analysis. This procedure permits the use of time signals recorded with the usual sampling frequency.

0 INTRODUCTION

The Wigner distribution has lately been put forward as a valuable tool for the characterization of the time-frequency behavior of filters, loudspeakers, and the like [1]–[3]. In most cases, for instance, when the time function to be transformed is the measured impulse response of a loudspeaker, use is to be made of the discrete Wigner distribution, whose properties were derived from the analytical Wigner distribution by Claasen and Mecklenbraeuker [4]. This discrete Wigner distribution requires a sampling frequency that is twice as high as in ordinary (time) signal analysis. This is due to the fact that the Wigner distribution involves the calculation of the function $f(t)\cdot f(t - a)$, which has twice the frequency contents of $f(t)$. This puts additional requirements on the measuring equipment, especially if the upper ranges of the audio band are to be studied. It also prohibits the analysis of previously recorded responses, when calculation of the discrete Wigner distribution was not yet considered.

There is, however, no real need for this high sampling frequency. It is possible to circumvent the problem through the use of a novel calculation procedure which makes it possible to keep the sampling frequency at its usual value and permits continued analysis of any previously recorded data.

1 MATHEMATICAL BACKGROUND

The well-known Shannon–Nyquist sampling theorem states that a signal $f(t)$ for which $|F(\omega)| = 0$ for $\omega >$ (sampling frequency/2) and is either limited to the interval $[-\pi, \pi]$ or periodical with a period time of 2π can in the interval $(-\pi, \pi)$ be represented exactly by the samples taken with this sampling frequency. In other words, if such a signal is represented by its Fourier series,

$$f(t) = \frac{a(0)}{2} + \sum_{n=1}^{N} a(n)\cdot\cos(n\cdot t) + b(n)\cdot\sin(n\cdot t) \qquad (1)$$

then $a(n)$ and $b(n)$ can be found exactly from the sampled version of $f(t)$. (The Fourier series can be truncated at N because all terms above N are zero due to the conditions mentioned.) However, this Fourier series holds for all values of t in the interval $(-\pi, \pi)$, not only at the instants of sampling. Hence it opens up the possibility to calculate the value of $f(t)$ for the instants halfway between two successive sampling times, thereby doubling the sampling frequency and making the time series thus obtained fit for analysis using the discrete Wigner distribution.

Basically, a reconstruction filter commonly used in digital audio systems does exactly the same thing.

Another possibility to circumvent the problem is to use the so-called "analytical" signal [2], because the contribution of the negative frequencies to this signal vanishes. However, to obtain this analytical signal, a procedure comparable to the sampling frequency doubling has to be followed. We will come back to this in the next section.

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2 CALCULATION PROCEDURE

Two almost identical calculation procedures can be followed to obtain this sampling frequency doubling. The most straightforward procedure uses a discrete Fourier transform, the other one a fast Fourier transform. Both start with calculating the Fourier series of $f(t)$, usually in its sampled counterpart written as $f(i \cdot dt)$, in which dt is the sampling interval, the inverse of the sampling frequency. With the discrete Fourier transform the value of $f(t)$ is then calculated using Eq. (1) for the desired values of t.

The calculation procedure using the fast Fourier transform requires the construction of a spectrum that has a range twice as wide as the original one. The low-frequency part of this spectrum should be filled with the values obtained from the Fourier transformation of the input signal; the high-frequency part must be filled with zeros. Inverse Fourier transformation of the spectrum thus created yields the desired time series with a doubled sampling frequency.

This procedure can also be understood in a spectral form. In Fig. 1 the spectrum of the original signal $f(t)$ is drawn in trace a. The spectrum of the sampled version of $f(t)$ is drawn in trace b. The periodicity that occurs in this spectrum limits reconstruction only at the sampling instants. After modification of the spectrum as described above it apears as shown in trace c. Such a spectrum with half the periodicity allows twice as many points in time to be reconstructed.

The analytical signal is a complex-valued time signal and can be obtained in a similar way. The first possibility is to multiply the Fourier series, obtained by a discrete Fourier transformation of the original time signal, by j, and to regard the time signal obtained from an inverse Fourier transform of the spectrum thus obtained as the imaginary part of the analytical signal, the original time signal being the real part of it. The second possibility is to replace the negative part of the spectrum obtained from a fast Fourier transform on the time signal with zeros, leave the dc term unchanged, and multiply the amplitude of the positive frequencies with 2. Inverse fast Fourier transformation on the spectrum thus obtained yields a complex-valued time signal, the analytical signal, with the same power spectrum as the original signal.

3 RESULTS

An example of this sampling frequency doubling is given in Fig. 2. The upper trace shows the input "signal," the lower trace the output "signal" of the sampling frequency doubling calculation procedure. The effect is obvious, especially at the first minimum.

4 CONCLUSION

The sampling frequency doubling calculation procedure reduces the need for a sampling frequency twice as high as usual for the calculation of the discrete Wigner distribution to a fairly straightforward calculation, which is short in comparison to the calculation of the discrete Wigner distribution itself. It is expected that this will stimulate the use of the discrete Wigner distribution since it takes away one of its drawbacks and opens up the possibility to use signals that have already been recorded with the usual sampling frequency.

5 REFERENCES

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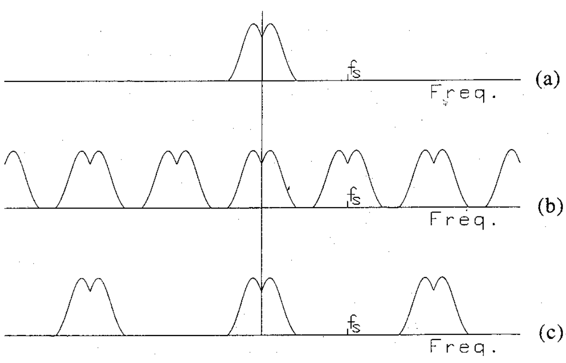

Fig. 1. Spectra of the signals studied. Trace a—original signal; trace b—sampled version; trace c—after modification for interpolation.

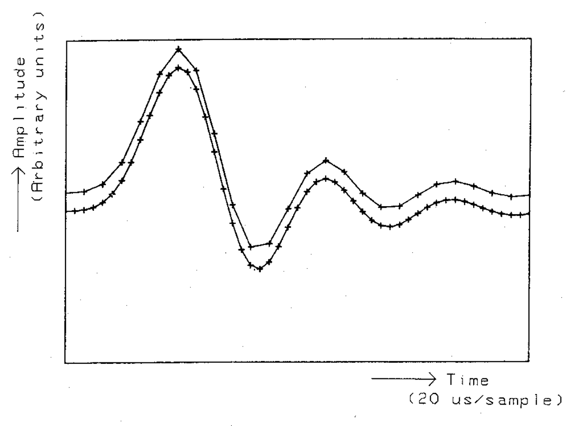

Fig. 2. Result of sampling frequency doubling procedure. Upper trace—input "signal;" lower trace—output "signal." Differences are obvious, but especially clear at first minimum.