Often disregarded Conditions for the correct Application of Fourier Theory

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Summary and Conclusions

Fourier theory is a powerful tool to predict the behaviour of audio systems, but the theory also imposes conditions which have to be fulfilled to obtain useful results. The most important ones are that the system should both be linear and time-invariant. Audio systems are neither, so the theory can only be approximately used. Therefore, careful application and interpretation of the results is required.

In this paper, the effect on the results when these conditions are not fulfilled will be illustrated, to illustrate the reasons behind these conditions. Unfortunately, no general rules for the effect of the approximations can be given, it needs to be verified for each individual case. But as neglecting these conditions will lead to incorrect results and conclusions, it is important to verify this always. It is also the basis of perceived differences between equipment with similar specifications as these do not take into account the response to music.

1. Introduction

The Fourier Theory is an elegant piece of mathematics, which describes how a signal in *time domain* can be transformed into a signal in *frequency domain*. In a bit more mathematical terms it says that the Fourier Transform (FT) converts a function from the time domain by a *one-to-one projection* into a function in the frequency domain. The "one-to-one" projection means that for each function in the time domain there is only one (and only *exactly* one!) corresponding function in the frequency domain. Which is why there is also the way back: the *Inverse Fourier Transform* (IFT) converts a function from the frequency domain by a one-to-one projection into the time domain. As a side remark it can be noted that the general Fourier theory discusses back and forth transforms of functions from the independent variable x to functions of the independent variable $1/x$. For more background on this theory, see textbooks like refs. 1 - 3.

In sec. 2, we will discuss the properties of the Fourier theory and how these can be used in audio, in sec. 3 its application to audio, in sec. 4, some properties of human hearing will be highlighted, in sec. 5, some of the consequences of non-confirmation to the requirements, leading to the Conclusions in sec. 6.

2. Properties of and requirements for the Fourier theory

The Fourier theory includes several very interesting and useful properties. One, which is of prime importance for audio, is that a signal of finite duration (which is always the case in reality) can be decomposed into an infinite series of sine and cosine waves of increasing frequency.

When the complex transfer function of a system, which will process this signal, is known, the output signal can be predicted. The way this works is as follows: the input signal is decomposed into the series of sine and cosine waves (preferably in complex number notation), yielding the input signal in the frequency domain. The signal (in frequency domain) is multiplied by the (complex) transfer function of the system, processing the signal, yielding the output signal in the frequency domain. An Inverse Fourier Transformation then yields the output signal in the time domain. This makes it very useful for all kinds of applications and phenomena in audio, e.g. to predict the output of a filter (in time domain) to an arbitrary input signal. Another well-known example is the upper frequency hearing limit: as humans cannot hear above 20 kHz, the reasoning is that there is no use in reproducing higher frequencies, as these will not contribute to the signal, reaching the brain. This argument has often been brought to the table to disqualify high-resolution audio. However, many high-end enthusiasts claim they can clearly hear the difference and even seniors, with an upper frequency upper limit of 10 kHz (like the author) can distinguish the difference. Also, the influence of reconstruction filters, which only act differently *above* 20 kHz, can be heard, even by seniors. Kunchur (refs. 4 and 5) has found that the temporal resolution of human hearing is far better than can be explained by the ability to hear frequencies using continuous tones. Understanding these findings requires a deeper look into the Fourier theory.

The *correct* application of the Fourier theory requires that several conditions are fulfilled. However, these are often disregarded, and two important conditions are that the system, to which it is applied, is *linear* (also internally!) and *time-invariant*.
It is not very hard to understand that these conditions are imposed: imagine that a signal is going through a system (whatever it may be), which distorts it. The output signal of this system therefore includes components which are *not present* in the input signal, like higher harmonics and intermodulation products. Filtering may *change* the relative amplitudes of the different components, but it *cannot introduce* novel components. As linear also requires that when the input signal is multiplied by two (2), the output signal also is <u>*exactly*</u> twice as large. In general, this is not the case with distortion.
To understand the time-invariance condition, a bit more thinking is required. As described above, the calculation of a response in time domain requires the multiplication in frequency domain of the input signal and the transfer function of the system. But when the properties of the system change in time, what or which 'transfer function' is to be used? And when the properties of the system are a function of the input signal due to memory effects? The deeper background is that Fourier theory is an 'integral' method, which already shows up by the way the representation of the signal in frequency domain is calculated: this requires integration over the entire interval of the signal. For more details, see refs. 1 – 3.

3. Applications of the Fourier theory in audio

When we look at electronics with active components, such systems are non-linear as has been described in a separate paper (Feedback Flaws). Which is why we have to deal with distortion. So, the first condition is, in general, not fulfilled. Memory effects also often occur in electronics, which can even be enhanced by non-linear effects. An example is shown in fig. 1:

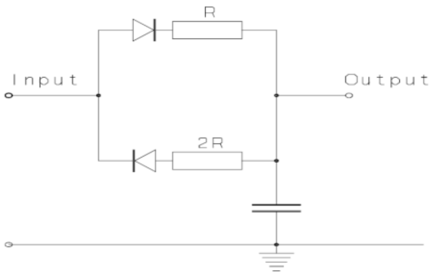

Figure 1: *Example of a circuit where the Fourier theory is not capable to predict the response, even if the continuous sine wave response would be known by measurement. The reasons are the non-linear behaviour (due to the diodes in the circuit) and the memory, created by the capacitor as its charge will depend on the input signal in the past.*

The determination of the frequency response of such a circuit is next to impossible (note that the dynamic impedance of the diodes depends on the current, flowing through them and is therefore also dependent on the amplitude of the input signal) and it is obvious that the system is highly non-linear. The charge on the capacitor will be a clear function of the history of the input signal, so the system is also not time-invariant. In other words, the application of Fourier theory to electronics is error-prone and there is a severe risk that the properties for continuous sine waves cannot (and will not) predict the response in time domain correctly. The non-linear properties are illustrated in figs. 2 – 5, which show the response in time of the circuit of fig. 1 to a more complex signal. The amplitude of the input signal is varied, the plots are scaled to the amplitude of the input signal. Blue is the input signal, yellow is the output signal. When the system would have been linear, the four figures would be indistinguishable, but this is obviously not the case. (**N.B.** The white, horizontal line represents zero Volts.)
To fulfil the requirement of linearity, one should keep in mind that this requirement has not only to be applied to the 'overall' system, but also internally, so e.g. also for the individual stages in an amplifier. One might think that because of a high global feedback factor, the amplifier is 'linear', but then the damage has already been done, as is explained in the paper on Feedback Flaws. All stages of an amplifier should be as linear as possible when Fourier theory is to be applied to *approximate* its response to music signals.

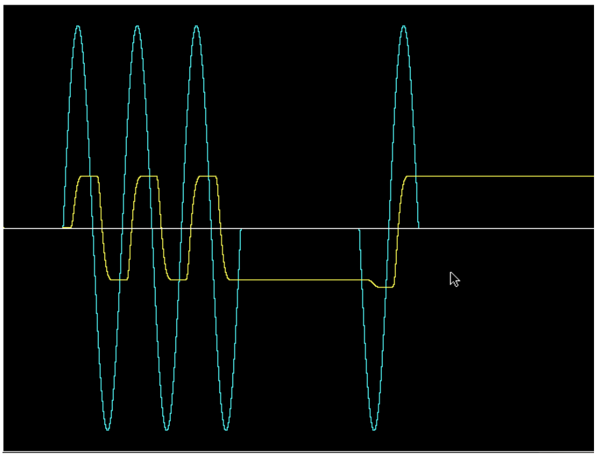

Figure 2: *Response in time domain (yellow) of the circuit of fig. 1 to a complex input signal (blue). Amplitude = 0.5 V*

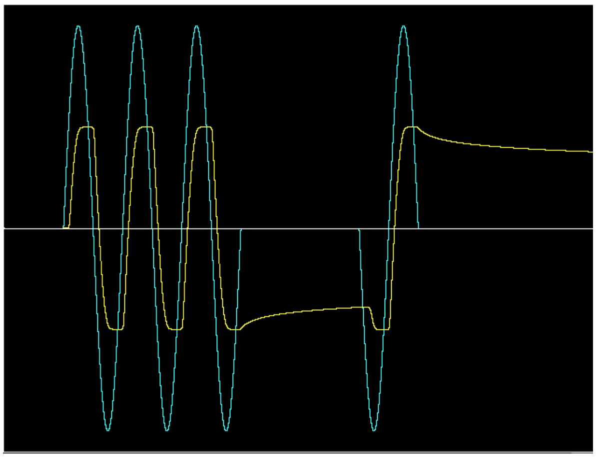

Figure 3: *Response in time domain (yellow) of the circuit of fig. 1 to a complex input signal (blue). Amplitude = 0.75 V*

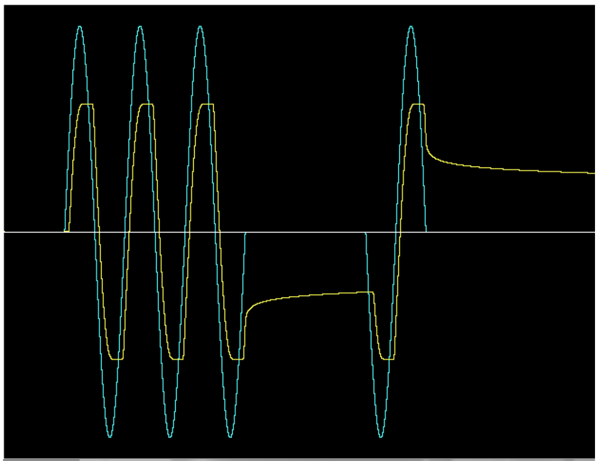

Figure 4: *Response in time domain (yellow) of the circuit of fig. 1 to a complex input signal (blue). Amplitude = 1 V*

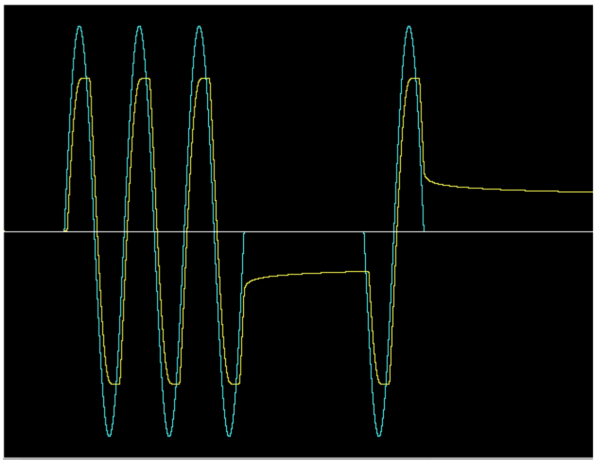

Figure 5: *Response in time domain (yellow) of the circuit of fig. 1 to a complex input signal (blue). Amplitude = 1.5 V*

The result of the memory effects are illustrated in figs 6 and 7, where the response to the second pulse is clearly influenced by the charge, left on the capacitor. If there would not have been a memory effect, the response to the second pulse in fig. 7 would be the mirror image of the response in fig. 6.

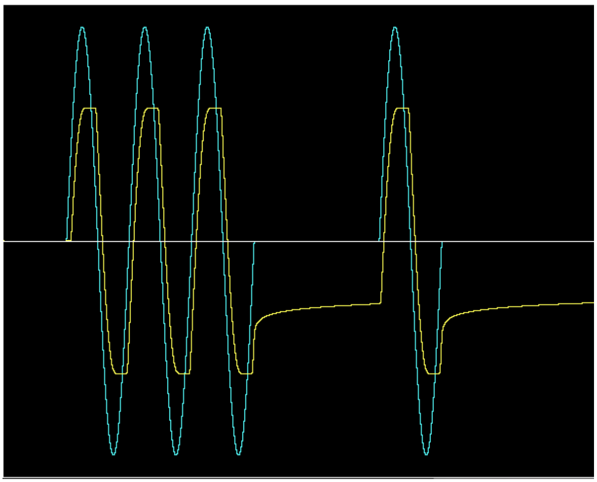

Figure 6: *Response in time domain (yellow) of the circuit of fig. 1 to a complex input signal (blue). Starting phase of second pulse is positive. Amplitude = 1 V*.

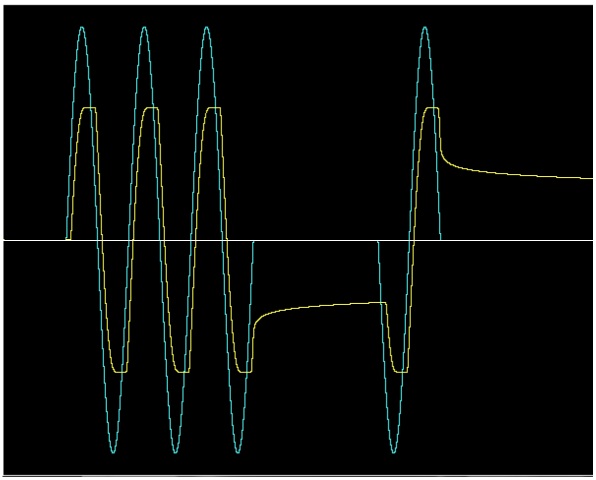

Figure 6: *Response in time domain (yellow) of the circuit of fig. 1 to a complex input signal (blue). Starting phase of second pulse is negative. Compare with fig.6. Amplitude = 1 V*.

When a system is not time-invariant, this usually means that the properties of the system depend in one way or another on what happened in the past. Memory effects can sneak into the design, which makes the behaviour of the amplifier dependant on the past input signal, even if overload is excluded. Components which are involved in memory effects often are capacitors and coils because these are able to store energy. Not surprisingly, these memory effects are mentioned in the paper on Feedback Flaws: the cut-off capacitor, the power supply and the loudspeaker loading. In combination with non-linear effects, the memory-effect can become more pronounced (see e.g. figs. 1, 6 and 7). During the design of high-end audio electronics, the potential memory-effects should be seriously considered, especially the parasitic ones, which easily escape attention.

The effects *in time domain* of non-linear behaviour in combination with memory effects could explain why e.g. amplifiers with similar properties regarding frequency response and distortion levels, sound different. It is to be expected that ten (10) different designs will produce ten different responses to music signals and thus receive a different perceptual qualification.

4. Some remarks about the non-linear properties of human hearing

Although it is outside the scope of this paper, it should be noted that human hearing is likely to be neither linear nor time-invariant, which can explain the findings of refs. 6 and 7. Another clear indication is that human hearing is able to circumvent the uncertainty relations (ref. 8). (**N.B.** Note that the uncertainty relations are a direct consequence of the *linear* Fourier Transformation). The findings of ref. 8 correspond to the findings of refs. 4 and 5. This could be the gateway to understand the discrepancy between the results, based on linear theory and the *perceptual* findings as described above. The often-remarkable properties of human hearing are the subject of a paper by Kunchur (to be published).

The temporal resolution of human hearing is at least an order of magnitude better than derived from its frequency response, so it is very likely that especially metal percussion instruments show a clear difference between 'live' and recorded sound. Alas, most microphones and tweeters are insufficiently at par with the temporal resolution of human hearing, so the perceived reproduction is clearly inferior to the 'live' sound. The 'high resolution audio' does improve the situation, but a major improvement of the microphones and tweeters is required to bring this to full fruition. More information on this subject can be found in several papers, which can be downloaded from www.temporalcoherence.nl

Although the Fourier theory has been well established since the second half of the 19th century, it is surprising that so little attention is given nowadays to the conditions, required to apply the *linear* theory. It has been applied unreluctantly to electronics and human hearing, even though neither fulfil either of these requirements. Therefore, it should not come as a surprise that the results are inconsistent with listening experiences. We have encountered that also in the paper on Feedback Flaws.

5. Consequences of non-compliance with the conditions of Fourier theory

It should be clear that when the conditions of linearity and time-invariance are not fulfilled, results, based on the Fourier theory, can be thrown straight into the wastepaper basket. Regretfully, these conditions are rarely respected and without hesitation, the frequency response, determined with continuous sinewaves, is interpreted as if it were from a linear and time-invariant system. Which explains why the behaviour of the amplifier with *dynamic* signals (like music) differs from the (expected) behaviour, based on results obtained with *steady, continuous* signals. To reproduce complex and dynamic signals like music well, the amplifier needs to be -next to a large number of other conditions- also as much as possible time-invariant and all its amplification stages should be as linear as possible. If not, artefacts will show up which manifest themselves mostly in the *time domain* and lead to a degradation of the sound stage and thus of the *perceived* quality. It is banging on an open door that the less an amplifier (also internally!) fulfils the requirements for a linear and time invariant system, the larger the contribution of artefacts to its output signal will be. As several of these cannot be detected using *continuous* sine waves, these differences may *not show up* in the specifications. This can explain why amplifiers with similar specifications give significant differences in the perceived quality.

6. Conclusions

The Fourier theory is very powerful and useful for audio, but it can only be applied correctly when the conditions imposed are fulfilled. The major requirements are linearity and time-invariance, but these are often not fulfilled, leading to incorrect results and conclusions. When the Fourier theory is used to predict the temporal properties of an audio system, one should realize that these conditions can only be approximately fulfilled. It should be verified to which extent the approximations will introduce deviations from the ideal, desired condition.

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